

Renormalized couplings and scaling correction amplitudes in the N -vector spin models on the sc and the bcc lattices

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For the classical N -vector model, with arbitrary N , we have computed through order β^{17} the high temperature expansions of the second field derivative of the susceptibility $\chi_4(N, \beta)$ on the simple cubic and on the body centered cubic lattices. (The N -vector model is also known as the $O(N)$ symmetric classical spin Heisenberg model or, in quantum field theory, as the lattice $O(N)$ nonlinear sigma model.) By analyzing the expansion of $\chi_4(N, \beta)$ on the two lattices, and by carefully allowing for the corrections to scaling, we obtain updated estimates of the critical parameters and more accurate tests of the hyperscaling relation $d\nu(N) + \gamma(N) - 2\Delta_4(N) = 0$ for a range of values of the spin dimensionality N , including $N = 0$ [the self-avoiding walk model], $N = 1$ [the Ising spin 1/2 model], $N = 2$ [the XY model], $N = 3$ [the classical Heisenberg model]. Using the recently extended series for the susceptibility and for the second correlation moment, we also compute the dimensionless renormalized four point coupling constants and some universal ratios of scaling correction amplitudes in fair agreement with recent renormalization group estimates.

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I. INTRODUCTION

We have recently extended the computation of high temperature (HT) series for the N -vector model¹ with arbitrary spin dimensionality N on the d -dimensional bipartite lattices, namely on the simple cubic (sc) lattice, on the body centered cubic (bcc) lattice and on their d -dimensional generalizations. In previous papers we have tabulated through order β^{21} the series for the zero field susceptibility $\chi(N, \beta)$ and for the second moment of the correlation function $\mu_2(N, \beta)$ and we have analyzed their critical behavior in the $d = 2$ case² and in the $d = 3$ case³. Here we present a study of the second field derivative of the susceptibility $\chi_4(N, \beta)$ whose HT expansion on the sc and the bcc lattices we have extended through order β^{17} . A study of $\chi_4(N, \beta)$ in the $d = 2$ case had been discussed in Ref.². It is interesting to point out that, in all analyses presented below, the bcc lattice series appear to be better converged than the sc lattice series and lead to estimates of critical parameters which are likely to be more accurate. In other words, the bcc series seem to be always "effectively longer"⁴ and therefore give estimates of greater value than the sc series.

The list of the expansions of $\chi_4(N, \beta)$ in $d = 3$ published up to now is a short one. A decade ago M. Lüscher and P. Weisz⁵ (see also Refs.⁶) derived HT expansions of $\chi_4(N, \beta)$ through β^{14} , for any N , on the sc lattice in $d = 2, 3$, and 4 dimensions by using a linked cluster expansion (LCE) technique^{5,7-12}. In the $N = 1$ case, [corresponding to the Ising spin 1/2 model] the series for the sc lattice published before our work already extended through $\beta^{17,13,14}$ and has been analyzed by various authors¹⁴⁻¹⁶. Finally in the Ising model case, a series to order β^{13} on the bcc lattice and a series to order β^{10} on the face centered cubic (fcc) lattice^{13,15,17} have long been available.

In our calculation we have also used the (vertex renormalized) LCE technique and have developed algorithms which are equally efficient in a wide range of space dimensionalities. So far other expansion methods have given competitive (or sometimes superior) performance only for discrete site variables and for very simple interactions, on two-dimensional or low coordination number lattices. By the LCE method we have produced tables of series expansion coefficients given as explicit functions of the spin dimensionality N , with an extension independent of the structure and dimensionality of the lattice. Thus we have succeeded in efficiently condensing a large body of information concerning infinitely many universality classes. We consider these coefficient tables to be the main result of our work and, in spite of their considerable extent, we have reported them in the appendix in order to make each step of our work verifiable and reproducible. The size of our computation has been unusually vast: approximately 3×10^6 topologically inequivalent graphs have been listed and evaluated. Nevertheless, we are confident that our series have been correctly computed, not only because our codes have been thoroughly tested, but also because N and d enter in the whole computational procedure as parameters. As a consequence, at least simple partial checks are available by observing that our expansion coefficients, when specialized to $N = 1$ agree with the series $O(\beta^{17})$ already available in 3 (as well as in 2) dimensions and, for $N \rightarrow \infty$, agree with the spherical model^{18,19} series which can be readily calculated in any dimension. More comments on the comparison of our results with the existing series, can be found in our paper² devoted to the two-dimensional N -vector model.

A valuable justification of our work is that an increasingly accurate study of the critical behavior of $\chi_4(N, \beta)$ can offer, for all values of N , a sharper test of the hyperscaling exponent relation $d\nu(N) + \gamma(N) - 2\Delta_4(N) = 0$. Here $\gamma(N)$ and $\nu(N)$ characterize the critical singularities in $\chi(N, \beta)$ and $\xi(N, \beta)$ respectively, while $\Delta_4(N)$ is the gap exponent associated with the critical behavior of the higher field derivatives of the free energy. It is also of great interest to measure accurately the critical amplitude of $\chi_4(N, \beta)$, which together with the amplitudes of $\chi(N, \beta)$ and $\xi(N, \beta)$, enters into the definition of the universal dimensionless renormalized four point coupling constant $g_r(N)$. Indeed the uncertainties, probably still of the order of 1%, in the value of this quantity might be the main residual source of error^{20,21} in the present computation of the critical exponents within the renormalization group (RG) approach by the Parisi²² fixed dimension (FD) coupling constant expansion^{23–27}. Murray and Nickel²⁰ have recently pushed to seven loop order these calculations and the impact of the additional terms on the estimates of the critical exponents and of some universal amplitude combinations has been critically assessed by Guida and Zinn-Justin²¹.

As has been stressed many times in the past two decades^{28–38} and, more recently, also in Ref.³, in order to improve the precision of the estimates obtainable from HT expansions not only longer series should be computed, but also more careful allowance should be made for the singular corrections to scaling. Their presence is expected³⁹ and, unsurprisingly, they turn out to be important in various cases. Therefore in this analysis we have also studied their role and have estimated their amplitudes in the case of $g_r(N)$, both on the sc and the bcc lattice. Moreover, it is of some interest to compute the ratios of these correction amplitudes with the analogous quantities for χ and ξ , which define interesting universal quantities, still subject to significant uncertainty and so far not much studied by HT series methods. We recall that most existing results on the universal combinations of the critical and the correction amplitudes are reviewed and thoroughly discussed in Refs.^{21,25,40}

The paper is organized as follows: In section 2 we present our notation and define the quantities we shall study. In section 3 we briefly discuss the simplified doubly biased differential approximants which we have used for our estimates beside more traditional numerical tools. Our analysis of the series is presented in section 4 along with a comparison to earlier series work, to measures performed in stochastic simulations and to RG estimates, both by the FD perturbative technique and by the Fisher-Wilson⁴¹ ϵ -expansion approach^{24–26,42,43}. Let us mention that, very recently, the ϵ -expansion of $g_r(N)$ has been extended by Pelissetto and Vicari⁴⁴ from order ϵ^{245} to order ϵ^4 , so that we are able to compare our HT results also with their estimates.

Our conclusions are briefly summarized in section 5. In the appendix we have reported the HT series coefficients of $\chi_4(N, \beta)$ expressed in closed form as functions of the spin dimensionality N . For convenience of the reader, we have also reported their evaluation for $N = 0$ [the SAW model⁴⁶], $N = 1$ [the Ising spin 1/2 model], $N = 2$ [the XY model] and $N = 3$ [the classical Heisenberg model]. The present tabulation supersedes and extends the one to order β^{14} in Ref.⁶ which, unfortunately, contains a few misprints.

II. DEFINITIONS AND NOTATION

We list here our definitions and notation. As the Hamiltonian H of the N -vector model we take:

$$H\{v\} = -\frac{1}{2} \sum_{\langle \vec{x}, \vec{x}' \rangle} v(\vec{x}) \cdot v(\vec{x}'). \quad (1)$$

where $v(\vec{x})$ is a N -component classical spin of unit length at the lattice site with position vector \vec{x} , and the sum extends to all nearest neighbor pairs of sites.

The susceptibility is defined by

$$\chi(N, \beta) = \sum_{\vec{x}} \langle v(0) \cdot v(\vec{x}) \rangle_c \quad (2)$$

where $\langle v(0) \cdot v(\vec{x}) \rangle_c$ is the connected correlation function between the spin at the origin and the spin at the site \vec{x} .

If we introduce the reduced inverse temperature $\tau^\#(N) = 1 - \beta/\beta_c^\#(N)$, (here and in what follows $\#$ stands for either sc or bcc, as appropriate, then $\chi(N, \beta)$ is expected to behave like

$$\chi^\#(N, \beta) \simeq C_\chi^\#(N) (\tau^\#(N))^{-\gamma(N)} \left(1 + a_\chi^\#(N) (\tau^\#(N))^{\theta(N)} + \dots + e_\chi^\#(N) \tau^\#(N) + \dots \right) \quad (3)$$

when $\tau^\#(N) \downarrow 0$. $C_\chi^\#(N)$ is the critical amplitude of the susceptibility, $a_\chi^\#(N)$ is the amplitude of the leading singular correction to scaling, $\theta(N)$ is the exponent of this correction (also called confluent singularity exponent) and $e_\chi^\#(N)$ is the amplitude of the leading regular correction. The dots represent higher order singular or analytic correction terms. The confluent terms result from the irrelevant variables³⁹. Let us recall that not only the critical

exponent $\gamma(N)$, but also the leading confluent correction exponent $\theta(N)$ is universal (for each N). On the other hand, the critical amplitudes $C_\chi^\#(N)$, $a_\chi^\#(N)$, $e_\chi^\#(N)$, etc. are expected to depend on the parameters of the Hamiltonian and on the lattice structure, i.e. they are non-universal. Similar considerations also apply to the other thermodynamic quantities listed below, which have different critical exponents and different critical amplitudes, but the same leading confluent exponent $\theta(N)$. It is known that $\theta(N) \simeq 0.5$ for small values of N^{25} and, in the context of the large N -expansion⁴⁷, one can also infer that $\theta(N) = 1 + O(1/N)$.

Since we have clearly stated which quantities are universal, from now on we shall generally omit the superscript $\#$ in order to keep the formulas more legible. Notice also that, since there is no chance of confusion, we have systematically omitted the superscript $+$ usually adopted for the amplitudes which characterize the high temperature side of the critical point.

The second moment of the correlation function is defined by

$$\mu_2(N, \beta) = \sum_{\vec{x}} \vec{x}^2 \langle v(0) \cdot v(\vec{x}) \rangle_c \quad (4)$$

In the vicinity of the critical point μ_2 is expected to behave as

$$\mu_2(N, \beta) \simeq C_\mu(N) \tau^{-\gamma(N)-2\nu(N)} \left(1 + a_\mu(N) \tau^{\theta(N)} + \dots + e_\mu(N) \tau + \dots \right) \quad (5)$$

as $\tau \downarrow 0$.

In terms of χ and μ_2 , the second moment correlation length ξ is defined⁴⁸ by

$$\xi^2(N, \beta) = \frac{\mu_2(N, \beta)}{6\chi(N, \beta)}. \quad (6)$$

In the vicinity of the critical point ξ is expected to behave as

$$\xi(N, \beta) \simeq C_\xi(N) \tau^{-\nu(N)} \left(1 + a_\xi(N) \tau^{\theta(N)} + \dots + e_\xi(N) \tau + \dots \right) \quad (7)$$

as $\tau \downarrow 0$.

The second field derivative of the susceptibility is defined by

$$\chi_4(N, \beta) = \frac{3N}{N+2} \sum_{x,y,z} \langle v(0) \cdot v(x) v(y) \cdot v(z) \rangle_c = \frac{3N}{N+2} \left(-\frac{2}{N} + \sum_{r=1}^{\infty} d_r(N) \beta^r \right). \quad (8)$$

Notice that this definition differs by a factor $1/N^2$ from that used in Ref.⁵.

It is well known^{18,19} that, for $N \rightarrow \infty$ at fixed $\tilde{\beta} \equiv \beta/N$, $\chi(N, \beta)$ has a finite non trivial limit $\tilde{\chi}(\tilde{\beta})$. On the other hand, as expected, in the same limit we have $\chi_4(N, \beta) = O(1/N)$. It is the quantity $N\chi_4(N, \beta)$ that has a finite limit $\tilde{\chi}_4(\tilde{\beta})$ simply expressed as

$$\tilde{\chi}_4(\tilde{\beta}) = -6\tilde{\chi}^2(\tilde{\beta})(\tilde{\chi}(\tilde{\beta}) + \tilde{\beta} \frac{d\tilde{\chi}(\tilde{\beta})}{d\tilde{\beta}}). \quad (9)$$

Also the $N \rightarrow 0$ limit, at fixed $\tilde{\beta}$, exists⁴⁶ and the quantity

$$\hat{\chi}_4(\tilde{\beta}) = \lim_{N \rightarrow 0} \chi_4(N, \beta) = -3 \sum_{N_1, N_2} c_{N_1 N_2} \tilde{\beta}^{N_1 + N_2} \quad (10)$$

has the following interpretation^{49,50}: $c_{N_1 N_2}$ is the number of pairs $(\omega^{(1)}, \omega^{(2)})$ of self avoiding walks such that $\omega^{(1)}$ is a N_1 -step walk starting at the origin and $\omega^{(2)}$ is a N_2 -step walk starting anywhere and crossing $\omega^{(1)}$.

In the vicinity of the critical point $\chi_4(N, \beta)$ is expected to behave as

$$\chi_4(N, \beta) \simeq C_4(N) \tau^{-\gamma(N)-2\Delta_4(N)} \left(1 + a_4(N) \tau^{\theta(N)} + \dots + e_4(N) \tau + \dots \right) \quad (11)$$

as $\tau \downarrow 0$.

In terms of χ , ξ and χ_4 the "dimensionless renormalized four point coupling constant" $g_r(N)$ is defined as the value of

$$g(N, \beta) \equiv -\frac{vf(N)\chi_4(N, \beta)}{\xi^3(N, \beta)\chi^2(N, \beta)} \quad (12)$$

at the critical point $\beta_c(N)$. Here $f(N) = \frac{N+8}{48\pi}$ is a normalization factor chosen in order to match the usual field theoretic definition of $g_r(N)$ ²⁵ and v denotes the volume per lattice site ($v = 1$ for the sc lattice and $v = 4/3\sqrt{3}$ for the bcc lattice).

In the vicinity of the critical point $g(N, \beta)$ is expected to behave as

$$g(N, \beta) \simeq g_r(N)\tau^{\gamma(N)+3\nu(N)-2\Delta_4(N)}\left(1 + a_g(N)\tau^{\theta(N)} + \dots + e_g(N)\tau + \dots\right) \quad (13)$$

as $\tau \downarrow 0$, with

$$g_r(N) = -\frac{vf(N)C_4(N)}{C_\xi^3(N)C_\chi^2(N)} \quad (14)$$

The Guntton-Buckingham⁵¹⁻⁵³ inequality

$$3\nu(N) + \gamma(N) - 2\Delta_4(N) \geq 0 \quad (15)$$

together with the Lebowitz⁵⁴ inequality $\chi_4(N, \beta) \leq 0$, implies that $g(N, \beta)$ is a bounded non-negative quantity as $\tau \downarrow 0$. The vanishing of $g(N, \beta_c)$ is a sufficient condition for Gaussian behavior at criticality, or, in lattice field theory language, for "triviality"⁴⁹ of the continuum field theory defined by the N -vector lattice model in the critical limit. If $\chi_4(N, \beta)$ is nonvanishing and the above inequality holds as an equality (the hyperscaling relation)

$$3\nu(N) + \gamma(N) - 2\Delta_4(N) = 0 \quad (16)$$

then

$$g(N, \beta) \simeq g_r(N)\left(1 + a_g(N)\tau^{\theta(N)} + \dots + e_g(N)\tau + \dots\right) \quad (17)$$

namely $g(N, \beta)$ tends to the nonzero limiting value $g_r(N)$ as $\tau \downarrow 0$.

For checking purposes it is useful to recall here the large N limits of the critical amplitudes. They have been computed¹⁹ long ago

$$C_\chi^{sc}(\infty) = \frac{1}{16\pi^2(\tilde{\beta}_c^{sc}(\infty))^3} = 0.39228768.. \quad (18)$$

with $\tilde{\beta}_c^{sc}(\infty) = 0.2527310098..$ and

$$C_\chi^{bcc}(\infty) = \frac{1}{64\pi^2(\tilde{\beta}_c^{bcc}(\infty))^3} = 0.29974101.. \quad (19)$$

with $\tilde{\beta}_c^{bcc}(\infty) = 0.1741504912..$

Moreover, we recall that, since in the large N limit

$$\tilde{\mu}_2 = q\tilde{\beta}\tilde{\chi}^2 \quad (20)$$

where q is the lattice coordination number, we have $C_\xi^\#(\infty) = (q\tilde{\beta}_c^\#(\infty)C_\chi^\#(\infty)/6)^{1/2}$. On the other hand, if we denote by $\tilde{C}_4^\#(\infty)$ the large N limit of $NC_4^\#(N)$, by (9) we have $\tilde{C}_4^\#(\infty) = -12(C_\chi^\#(\infty))^3$ and therefore it follows that $g_r^\#(\infty) = 1$.

III. ANALYSIS OF THE SERIES

As mentioned in the introduction, a variety of careful analyses²⁸⁻³⁸ of the Ising model HT expansions as well as our study of the recently extended N -vector model series³, suggest that the non-analytic confluent corrections to the leading critical behavior of the thermodynamic quantities, indicated in the asymptotic formulas(3),(5), (7), etc. exist and should not in general be neglected in computing numerical estimates of the critical parameters. It has

long been observed^{28–37,55} that these corrections reveal themselves as small apparent violations of both universality and hyperscaling in a naive pure power law analysis of the critical behavior. However it is also well known^{31,32} that, unless very long HT series are available, extracting simultaneously estimates for β_c , the exponents and the amplitudes of the critical *and* of the subleading singularity is a difficult and unstable numerical problem. For this task the inhomogeneous DA method⁵⁶ of series analysis is generally believed to be more effective than the traditional and simpler Padé approximant (PA) method, because, at least in principle, it might be flexible enough to represent functions behaving like $\phi_1(x)(x-x_0)^{-\omega} + \phi_2(x)$ near a singular point x_0 , where $\phi_1(x)$ is a regular function of x and $\phi_2(x)$ may contain a (confluent) singularity of strength smaller than ω . Unfortunately, in practice, this is not completely true: very long series are needed anyway and/or the procedure should be biased by choosing very carefully the structure of the approximants and by giving proper inputs. We have followed here the latter approach. As in some of our previous studies^{3,57}, beside more standard procedures of analysis, we have employed a doubly biased prescription which assumes that the confluent exponent θ and the inverse critical temperature β_c are accurately known. This procedure seems to perform reasonably well, even with not very long series. We have taken the values of $\theta(N)$ as estimated by the FD renormalization group method. More precisely, for $N \leq 4$, we have used the values suggested by Guida and Zinn-Justin²¹, and for $N > 4$, we have used the six loop estimates recently obtained by A. I. Sokolov⁵⁸ and kindly communicated to us before publication. These values are reported in table 1. We also have assumed that the critical temperatures $\beta_c^\#(N)$ have been determined accurately enough in our previous study of the susceptibility³.

Let us now recall in some detail the features of the simplified DA method.

We wish to approximate some function, given as a series expansion around $\beta = 0$ and expected, when $\beta \uparrow \beta_c$, to have the form

$$f(\beta) = \sum_{n=0} f_n \beta^n \simeq b(\beta) + c(\beta)(1 - \beta/\beta_c)^\theta + o((1 - \beta/\beta_c)^\theta). \quad (21)$$

We assume that β_c and the real positive exponent θ are accurately known, and that $b(\beta)$ and $c(\beta)$ are analytic at $\beta = \beta_c$. We set $b(\beta_c) = b_0$ and $c(\beta_c) = c_0$.

We shall estimate the function $f(\beta)$ and therefore the constants b_0 and c_0 by the particular class of first order inhomogeneous differential approximants $F(\beta)$ defined as the solutions of the equations

$$Q_m(\beta) \left[(1 - \beta/\beta_c) \frac{dF(\beta)}{d\beta} + \frac{\theta}{\beta_c} F(\beta) \right] + R_n(\beta) = 0 \quad (22)$$

with the initial condition $F(0) = f_0$.

$Q_m(\beta)$ and $R_n(\beta)$ are polynomials of degrees m and n respectively, whose coefficients are calculated, as usual, by imposing that the series expansion of $F(\beta)$ agrees with that of $f(\beta)$ at least through the order β^{m+n+1} . In addition the normalization condition $Q_m(0) = 1$ is imposed. Assuming for simplicity $0 < \theta < 1$, $f(\beta_c) = b_0$ is estimated as

$$b_0^{(n,m)} = \frac{-\beta_c R_n(\beta_c)}{\theta Q_m(\beta_c)} \quad (23)$$

and the amplitude c_0 of the subleading term in Eq. (21) is estimated by the formula

$$c_0^{(n,m)} = f_0 - b_0^{(n,m)} - \int_0^{\beta_c} \frac{D^{(n,m)}(t) dt}{(1 - t/\beta_c)^{1+\theta}} \quad (24)$$

where

$$D^{(n,m)}(t) = \frac{R_n(t)}{Q_m(t)} - \frac{R_n(\beta_c)}{Q_m(\beta_c)} \quad (25)$$

We shall consider only the "almost diagonal" approximants with $|m - n| \leq 4$.

The approximants defined by (22) are just a small subclass of the general first order inhomogeneous DA's

$$(1 - \beta/\beta_c) Q_m(\beta) \frac{dF(\beta)}{d\beta} + P_l(\beta) F + R_n(\beta) = 0 \quad (26)$$

biased with β_c and with θ by imposing $P_l(\beta_c)/Q_m(\beta_c) = \frac{\theta}{\beta_c}$. Still assuming $0 < \theta < 1$, we can estimate b_0 and c_0 from (26) as follows

$$b_0^{(m;n,l)} = -\frac{R_n(\beta_c)}{P_l(\beta_c)} = -\frac{\beta_c R_n(\beta_c)}{\theta Q_m(\beta_c)} \quad (27)$$

$$c_0^{(m;n,l)} = -b_0^{(m;n,l)} + g^{(m;n,l)}(\beta_c) \left[f_0 - \int_0^{\beta_c} \frac{D^{(m;n,l)}(t)}{g^{(m;n,l)}(t)(1-t/\beta_c)^{1+\theta}} dt + \frac{\theta}{\beta_c} b_0^{(m;n,l)} \int_0^{\beta_c} \left(\frac{1}{g^{(m;n,l)}(t)} - \frac{1}{g^{(m;n,l)}(\beta_c)} \right) \frac{dt}{(1-t/\beta_c)^{1+\theta}} \right] \quad (28)$$

where

$$g^{(m;n,l)}(\beta) = \exp \left[- \int_0^\beta \left(\frac{P_l(t)}{Q_m(t)} - \frac{P_l(\beta_c)}{Q_m(\beta_c)} \right) \frac{dt}{(1-t/\beta_c)} \right] \quad (29)$$

and $D^{(m;n,l)}(t)$ has the same form as (25). The simple formulas (23) and (24) are recovered from the general formulas (27) and (28) by subjecting $P_l(\beta)$ to the further strong constraint $P_l(\beta) \equiv \frac{\theta}{\beta_c} Q_m(\beta)$. This prescription, which, for short, we will refer to as simplified differential approximants (SDA's) might also be viewed as a simple DA-like generalization of the biased PA method introduced in Refs.⁵⁹⁻⁶¹.

We have carried out many numerical experiments on simple model series having the analytic structure (21). They show that the SDA's, when biased with the exact values of β_c and θ , are able to produce very accurate estimates of b_0 and fairly accurate estimates of the confluent amplitude c_0 . In practice however, we do not have strict control on the series: only approximate values of β_c and θ are available for biasing the SDA's and we do not know the strength of the subleading correction terms and of the smooth background. Therefore it is important to understand how sensitive are the estimates of b_0 and c_0 to the errors in the biased inputs and how they depend on the structure of the singularity. It turns out that the estimates of b_0 are rather stable when the biased value for β_c and for θ are varied away from their true values in a range comparable to the typical estimated uncertainties in the realistic cases. On the other hand, c_0 appears to be much more sensitive to errors in the biased values. Let us consider, to be definite, the case of the very simple test series

$$f(\beta) = c_0 \left(1 - \frac{\beta}{\beta_c}\right)^\theta + c_1 \left(1 - \frac{\beta}{\beta_c}\right)^{2\theta} + b_0 \exp\left(1 - \frac{\beta}{\beta_c}\right) \simeq b_0 + c_0 \left(1 - \frac{\beta}{\beta_c}\right)^\theta + o((\beta_c - \beta)^\theta) \quad (30)$$

which we have examined for various values of θ . If the size of the subleading correction to scaling is much smaller than the size of the leading one, namely if $|c_1| \ll |c_0|$ and we bias the calculation with the exact values of the parameters θ and β_c , we are able to determine b_0 by (23) to within less than $10^{-2}\%$ and c_0 by (24) to within less than 1%. However, if the SDA's are biased with a value of θ which is off the right value by 5%, then the relative error of c_0 can become as large as 15%, while the error of b_0 increases to some 0.1%. The precision of b_0 remains essentially unchanged, but the sensitivity of c_0 to variations in the biased values and, as a consequence, the accuracy of its estimate is somewhat worsened in the slightly more complicated, but sometimes realistic case in which $|c_1| \approx |c_0|$. Unsurprisingly, the worst situation occurs when the leading confluent amplitude is much smaller than the subleading one, since the uncertainty in the numerical estimate of c_0 may then become very large. In conclusion, taking a conservative attitude, we can safely expect that, for the HT series we are going to study, the relative error on the value of $f(\beta)$ at β_c can be much smaller than 1%, while the uncertainty of the correction amplitude can be as large as 20%, unless the amplitude is very small: in this case, due to a higher sensitivity to the biased values and/or to the neglect of possibly important subleading corrections, our estimates are likely to be much more inaccurate. In order to better understand these results let us also observe that, if we tried to estimate b_0 in (30) by simple PA's biased with β_c , the relative error would be substantially larger and increasing with the size of the correction amplitude. Finally, we remark that in all computations presented below, the error estimates are always somewhat subjective. They include effects both from the scatter of the approximant values, possible residual trends in sequences of estimates, as well uncertainties of the bias inputs.

We have applied the SDA approximation procedure not only to the quantity $g(N, \beta)$ in order to compute the confluent amplitude $a_g(N)$, but also to the "effective exponent" of χ_4

$$\gamma_4(N, \beta) \equiv (\beta_c(N) - \beta) \frac{d \log \chi_4(N, \beta)}{d\beta} = \gamma(N) + 2\Delta_4(N) - a_4(N) \theta(N) \tau^{\theta(N)} + o(\tau^{\theta(N)}) \quad (31)$$

in order to compute the critical exponent and the confluent amplitude $a_4(N)$.

Moreover we have examined the analogous quantities

$$\gamma(N, \beta) \equiv (\beta_c(N) - \beta) \frac{d \log \chi(N, \beta)}{d\beta} = \gamma(N) - a_\chi(N) \theta(N) \tau^{\theta(N)} + o(\tau^{\theta(N)}) \quad (32)$$

in order to compute the confluent amplitude $a_\chi(N)$, and

$$\nu(\beta, N) \equiv \frac{1}{2}(\beta_c(N) - \beta) \frac{d \log [\xi^2(N, \beta)/\beta]}{d\beta} = \nu(N) - a_\xi(N) \theta(N) \tau^{\theta(N)} + o(\tau^{\theta(N)}) \quad (33)$$

in order to compute $a_\xi(N)$. Notice that the estimates thus obtained for the confluent amplitudes a_χ , a_ξ , and a_4 are biased solely with β_c and θ . However, due to their definition as residua, the sensitivity of the results to the biased value for β_c is higher than in the case of g_r .

The estimates of the critical amplitudes have been obtained by examining quantities like

$$\tau^{\gamma(N)} \chi(N, \beta) \simeq C_\chi(N) \left(1 + a_\chi(N) \tau^{\theta(N)} + \dots + e_\chi(N) \tau + \dots \right) \quad (34)$$

or the analogous expressions for χ_4 and ξ^2 . This procedure also yields the correction amplitudes, but since it requires biasing also with the critical exponent $\gamma(N)$ (or $\nu(N)$ etc.), we expect that the corresponding results will be subject to a larger uncertainty.

In conclusion, whenever sizable confluent corrections are present, the doubly biased SDA procedure will produce values of $g_r(N)$ which are slightly, but definitely different from estimates by generic DA's not directly constrained to reproduce the confluent singularity and, *a fortiori*, from the simple PA estimates. Indeed, since $\theta < 1$, the function $g(N, \beta)$ will approach with a divergent slope its value at $\beta_c(N)$, from above if the correction amplitude is positive or, otherwise, from below. As a consequence, too smooth extrapolations of $g(N, \beta)$ to the critical point β_c would overestimate the correct result in the former case and underestimate it in the latter. Analogous problems will occur in the study of the exponents and of the correction amplitudes for χ, μ_2, χ_4 , the only difference being that, since in the formulas for the effective exponents (31), (32) and (33) the correction amplitudes appear with a negative sign, the critical exponents will be overestimated if the amplitudes are negative and they will be underestimated otherwise.

Let us add finally that throughout our work we have not relied solely on the above numerical technique, but we also have always considered various other approximations obtained by more conventional methods in order to understand, or at least to be aware of any differences in the estimates.

IV. RESULTS AND COMMENTS

Since our analysis is aimed at exposing the role of the non analytic corrections to scaling, it is desirable firstly to test whether the values of the confluent exponents taken from the FD perturbative computations are also generally consistent with the estimates, unfortunately not yet as precise, which can be extracted directly from the HT series. Indeed, as we have mentioned above, the amplitudes of these corrections are not universal and therefore they might be negligibly small. One might even suspect that our analysis is somehow artificially forcing on the series a behavior, which, due to their insufficient length, they are not yet able to exhibit. On the other hand, it has been argued that the uncertainties usually quoted for the FD values of the renormalized couplings and of the confluent exponents might be unrealistically small^{31,44,50,62}. In fact, one should recall that in the context of the three-dimensional $\lambda(\vec{\phi}^2)^2$ field theory, the confluent exponent is computed in terms of the slope of the beta-function at the fixed point $g_r(N)$. As indicated in Refs.^{21,31,44}, the presence of non-analytic terms, with sufficiently large amplitudes, in the expansion of the beta-function at $g_r(N)$, might spoil the convergence of the estimates both of the renormalized couplings and of the confluent exponents. The ensuing uncertainties would reflect on the accuracy of the estimates of the critical exponents. Moreover the g -expansion of the critical exponents would itself be directly affected by similar non-analytic contributions. The pragmatic point of view adopted in Ref.²¹ is that if these singular terms exist, they do not seem to have visible effects.

Let us then show that the values of $\theta(N)$ reported in Table 1 are approximately consistent with the actual behavior of the series. Assuming knowledge only of $\beta_c^\#(N)$, we have computed the Baker-Hunter transforms⁶³ of the χ and μ_2 series and, by reconstructing the locations and the residua of their singularities, we have estimated exponents and amplitudes of the critical singularity and of the leading correction to scaling. Unfortunately this procedure fails to detect narrow and clear signals of the scaling corrections for $N < 4$, probably due to the small size of their amplitudes. But the situation is completely different for $N \geq 4$. In this range of values of N , the Baker-Hunter method leads to values of $\theta(N)$ fairly consistent with those reported in Table 1. Also the values of the correction amplitudes, are compatible with those emerging from the SDA analysis to be discussed below. Moreover, the results are rather stable in a relatively wide range of biased values for β_c . We regard this as convincing evidence that the confluent corrections

cannot be a by-product of our double biased analysis and as a confirmation that their amplitudes are not small for $N \geq 4$. Unfortunately, the uncertainties which affect this method for estimating the confluent exponents and the correction amplitudes are still rather large. For instance, using the bcc lattice series for χ , the Hunter-Baker procedure suggests

$$\theta(4) = .64(4); \quad \theta(6) = .63(4); \quad \theta(8) = .66(4); \quad \theta(10) = .69(4) \quad (35)$$

A second consistency test can also be performed. On both lattices and for each value of N , we have studied how our SDA estimates of $g_r(N)$ depend on the biased value used for the confluent exponent by varying it in a 20 – 30% range around the central value $\theta(N)$ indicated in Table 1. For all values of N such that the confluent amplitudes are not too small, it has been quite interesting to observe that, although the estimates of $g_r(N)$ obtained from the sc and the bcc series are in general somewhat different for a generic value of θ , they tend to become equal, or at least very close, when $\theta \simeq \theta(N)$.

These two tests give us further confidence that the main lines of this analysis and the specific biased values of θ used as inputs are reasonable.

A. Hyperscaling tests

We shall now proceed to examine directly $\chi_4(N, \beta)$ in order to estimate its critical exponent $\gamma(N) + 2\Delta_4(N)$ and to compare it with the value $2\gamma(N) + 3\nu(N)$ it should take if the hyperscaling relation (16) holds true. On each lattice, the analysis has been performed by first-order SDA's of the effective exponent (31) doubly biased with $\theta(N)$ and with the value of $\beta_c^\#(N)$ obtained in our previous (biased) analysis³ of the susceptibility.

We have reported in table 2 our estimates for the critical exponents of $\chi_4(N, \beta)$ obtained by this procedure together with the biased values of $\beta_c^\#(N)$ and the values of $2\gamma(N) + 3\nu(N)$ obtained by the analogous biasing procedure in our previous analysis³ of χ and ξ^2 . No significant violation of universality and hyperscaling is observed. Notice that no such extensive test of hyperscaling exists so far in the literature.

Let us quote a few earlier studies of this issue for particular values of N . In the $N = 0$ case, a study of χ_4 based on (10) has been performed by a Monte Carlo simulation in Ref.⁶⁴. The authors have measured the exponents $2\Delta_4 - \gamma = 1.7317 \pm 0.0074 \pm 0.0074$ and $\nu = 0.5745 \pm 0.0087 \pm 0.0056$. The final result is expressed as $3\nu + \gamma - 2\Delta_4 = -0.0082 \pm 0.0027 \pm 0.018$, the first error being systematic and the second statistical.

In the $N = 1$ case, the tests of the hyperscaling relation (16) are numerous and have a long history^{35,65}. The validity of (16) for the 3d Ising model had been questioned by G. A. Baker^{9,66} on the basis of an analysis of 10-12 term series for the sc, bcc and fcc lattices. A few years later, when B. G. Nickel computed $O(\beta^{21})$ series on the bcc lattice for χ and μ_2 in the spin S Ising model, it became clear that rather long series were necessary to allow for the scaling corrections and thus to obtain more satisfactory estimates of γ and ν ^{31–33,36}. On the other hand accurate analyses of the critical behavior of the $\chi_4(1, \beta)$ series to order β^{17} on the sc lattice^{14,15,57,67} had yielded reliable values also for $\Delta_4(1)$. On the basis of these results, as well as of various recent Monte Carlo results^{68–70} a common consensus was reached that, for $N = 1$, if any violation of (16) occurs, it should be much smaller than was initially suspected. Our contribution to this issue also consists in providing an extension from order β^{13} to order β^{17} of the Ising bcc series for χ_4 , and therefore in further improving the accuracy of the HT test of hyperscaling and universality even for the widely studied $N = 1$ case.

B. Renormalized couplings

Let us first mention that, since $\xi^2 = O(\beta)$ in the vicinity of $\beta = 0$, from the series for χ , ξ^2 and χ_4 we can form two distinct auxiliary functions $w(N, \beta)$ and $u(N, \beta)$, analytic at $\beta = 0$, both of which, when extrapolated at β_c yield $g(N, \beta_c)$ and therefore $g_r(N)$, if we assume the validity of the hyperscaling relation. More precisely we shall consider:

$$u(N, \beta) \equiv -\frac{\xi^2(N, \beta)\chi^{4/3}(N, \beta)}{(vf(N)\chi_4(N, \beta))^{2/3}} \quad (36)$$

whose value at $\beta_c(N)$ is $g_r(N)^{-2/3}$, and

$$w(N, \beta) \equiv -\frac{vf(N)\chi_4(N, \beta)}{\beta_c^{3/2}(\xi^2(N, \beta)/\beta)^{3/2}\chi^2(N, \beta)} \quad (37)$$

whose value at $\beta_c(N)$ is $g_r(N)$.

It is interesting to form approximants both of $u(\beta)$ and of $w(\beta)$ because for various values of N , at the presently available order of expansion, they still show slightly different convergence properties. This may be seen as an indication that the χ_4 series are still not very long. Indeed, as we have argued in Ref.⁵⁷, at order β^s the dominant contributions to the HT expansion of χ_4 come from correlation functions of spins whose average distance is $\approx s/4$. Therefore the presently available expansions with $s_{max} = 17$ still describe only a rather small system.

Table 3 contains our estimates of the universal renormalized coupling $g_r(N)$.

For $N \leq 4$ we have evaluated $g_r(N)$ by forming SDA's of the auxiliary function $w(N, \beta)$, which has been chosen because it yields sequences of estimates showing little or no residual trends when an increasing number of series coefficients is used. On the other hand, for $N > 4$, we have used $u(N, \beta)$ because the estimates obtained from it show the slowest (generally decreasing) residual trends. Whenever relevant, we have indicated this fact by reporting asymmetric error bars.

In the $N = 0$ case, allowance for the correction to scaling yields a value of $g_r(0)$ approximately 2% smaller than the one recently obtained within the FD expansion²¹, but very close to the value suggested⁴⁴ by the ϵ -expansion. Our value is also close to that indicated in Ref.²⁰ and produces, via the seven loop FD perturbation series, central values of $\gamma(0)$ and $\nu(0) \approx 0.2\%$ lower than those quoted in Ref.²¹, but within their error bars. It is also worth recalling that also our earlier HT analysis³ of $\chi(0, \beta)$ and $\xi^2(0, \beta)$ had supported those low exponent estimates in good agreement with very recent high precision measures by stochastic methods on the sc lattice⁷¹.

For $N = 1$, on the sc lattice, we have reported here a central estimate of $g_r(1)$ slightly lower than, though consistent with the estimate $g_r(1) = 1.411$ obtained from our previous analysis⁵⁷ based on SDA's of $u(1, \beta)$, rather than of $w(1, \beta)$.

A small sample of the most recent estimates of $g_r(1)$ by various methods has also been included in the table. All of them appear to be mutually consistent, if we consider how difficult it has been to achieve very accurate Monte Carlo measures of $g_r^{sc}(1)$ ^{68–70} and we recall that, even in the Ising case, the previous HT series estimates⁷² of the renormalized coupling were based on expansions shorter than those presented here. Indeed, although $\chi_4(1, \beta)$ on the sc lattice has long been known through order β^{17} , the corresponding expansion for the renormalized coupling was not available before our recent work^{3,57}, because $\xi^2(1, \beta)$ reached¹⁶ only order β^{15} . On the other hand the $\chi_4(1, \beta)$ series for the bcc lattice was known to order β^{13} only.

To our knowledge, no Monte Carlo results are yet available for $N > 1$.

For $N \geq 3$ our estimates are systematically slightly higher than the FD values of Refs.^{21,58}, and perhaps the residual decreasing trend in our estimates might not be sufficient to reconcile them. This difference is related to our allowance of the scaling corrections by doubly biased SDA's and is consistent with the higher values of γ and ν that we had obtained in our biased analysis³ of χ and ξ . As we have stated above in discussing the general features of the SDA's, significantly larger estimates for $N < 4$ and somewhat lower estimates for $N \geq 4$ would be obtained, if the renormalized couplings were evaluated by simple PA's. This fact is completely consistent with the observed behavior of the correction amplitudes as functions of N to be discussed in next subsection. A similar observation has been made also in Ref.⁴⁴ where, on the basis of the old sc lattice $O(\beta^{14})$ series^{5,6}, the $g_r(N)$ have been evaluated by ordinary DA's, either directly or after performing a change of variable^{59–61} designed to regularize the leading correction to scaling and numerically similar to our SDA's. Therefore the final HT estimates of Ref.⁴⁴ essentially agree with ours.

We have included in table 3 some estimates of $g_r(N)$ based on the ϵ -expansion to order ϵ^4 recently presented in Ref.⁴⁴. They are compatible with ours for $N < 3$, while, for $N \geq 3$, the central values are $\approx 2\%$ lower.

C. Critical and correction amplitudes

In tables 4 and 5 we have reported our estimates of the (non-universal) critical amplitudes $C_\chi^{sc}(N)$, $C_\xi^{sc}(N)$ and $C_\chi^{bcc}(N)$, $C_\xi^{bcc}(N)$, based on the values of $\beta_c^\#(N)$, $\gamma^\#(N)$ and $\nu^\#(N)$ obtained in the biased analysis of Ref.³. Earlier determinations of the critical amplitudes either from the extrapolation of (generally shorter) HT series or from stochastic simulations are also available for $N = 1$ in Refs.^{72,73}, for $N = 2$ in Ref.⁷⁴, and for $N = 3$ in Ref.⁷⁵. However, comparisons with the results of tables 4 and 5, which in general are close to the earlier ones, are not very illuminating, since the estimates depend sensitively on the numerical procedures, on the biased values used for $\beta_c^\#(N)$ and on the relevant critical exponents, which are slightly different in the various studies.

For example in the $N = 1$ case^{37,72} (on the basis of shorter sc lattice series, but the same bcc series), the following estimates are proposed: $C_\chi^{sc}(1) = 1.0928(10)$ and $C_\xi^{sc}(1) = 0.4984_{(50)}^{(10)}$; $C_\chi^{bcc}(1) = 1.0216(8)$ and $C_\xi^{bcc}(1) = 0.4608(2)$ assuming $\beta_c^{sc}(1) = 0.221630(12)$, $\beta_c^{bcc}(1) = 0.157368(7)$, $\gamma(1) = 1.2395$ and $\nu(1) = 0.632$.

For $N = 2$, in Ref.⁷⁴ the estimates $C_\chi^{sc}(2) = 1.058(7)$ and $C_\xi^{sc}(2) = 0.498(2)$ have been obtained from a fit to Monte Carlo data, assuming $\gamma(2) = 1.3160(25)$, $\nu(2) = 0.669(2)$, allowing for confluent corrections with exponent

$\theta(2) = 0.522$. This fit also yields $\beta_c^{sc}(2) = 0.454162(9)$ from the analysis of χ , and $\beta_c^{sc}(2) = 0.454167(10)$, from the analysis of ξ .

For $N = 3$, in Ref.⁷⁵, the estimates $C_\chi^{sc}(3) = 0.955(6)$ and $C_\xi^{sc}(3) = 0.484(2)$ have been obtained from a fit to Monte Carlo data (with no allowance for confluent corrections) also yielding $\beta_c^{sc}(3) = 0.69294(3)$, $\gamma(3) = 1.391(3)$ from the analysis of χ and $\beta_c^{sc}(3) = 0.69281(4)$, $\nu(3) = 0.698(2)$, from the analysis of ξ .

As has been stressed in general in Ref.³⁸ and as we have anticipated in our considerations of Section 3 on the numerical properties of SDA's, the discussion of the estimates of the scaling correction amplitudes is much more delicate. Let us first comment on some qualitative features of the estimates of these amplitudes for the sc and the bcc lattices which are denoted as $a_\chi^{sc}(N)$, $a_\chi^{bcc}(N)$, $a_\xi^{sc}(N)$, $a_\xi^{bcc}(N)$, $a_g^{sc}(N)$, $a_g^{bcc}(N)$, and reported in table 6.

Both correction amplitudes $a_\chi^\#(N)$ and $a_\xi^\#(N)$ are negative for $N \lesssim 2$, whereas they are positive and increasing for $N > 2$. (Actually we have reported a positive value for $a_\chi^{bcc}(2)$, but with a large uncertainty.) Therefore the ratio a_ξ/a_χ is very likely to be positive for all values of N . The correction amplitudes $a_\chi^\#(N)$ and $a_\xi^\#(N)$ turn out to be small, but not negligible for $N \leq 1$ and rather large for $N \geq 4$, both in the sc and in the bcc lattice case. On the contrary, for $N = 2$ and $N = 3$ they are very small. Thus the overall behavior of the correction amplitudes for χ and ξ as functions of N appears to be smooth and completely consistent with the size and the sign of the differences³ between our unbiased estimates of the critical exponents γ and ν and the corresponding estimates biased with both β_c and θ . More precisely, we recall that the non-analytic corrections to scaling lead to (slightly) higher effective exponents for $N < 2$ and to (significantly) lower effective exponents for $N \geq 4$. On the other hand, the $a_g^\#(N)$'s are positive for $N \lesssim 2$, while they are negative and decreasing for $N > 2$, so that the ratio a_g/a_χ is negative for any N . (Actually we have reported a positive value for $a_g^{sc}(3)$, but with a large uncertainty.) The $a_g^\#(N)$'s are generally not small, except for $N = 3$ in the sc case and for $N = 2$ in the bcc case.

It is appropriate now to quote some earlier evaluations of a_χ and a_ξ by HT series or Monte Carlo simulations. In the $N = 1$ case, it had been established^{35,36,38,76} long ago that the sign of $a_\chi(1)$ and of $a_\xi(1)$ is negative on the sc, bcc and fcc lattices. In Ref.³⁶, for the spin 1/2 Ising model on the bcc lattice, the estimates $a_\chi^{bcc}(1) \approx -0.13$ and $a_\xi^{bcc}(1) \approx -0.11$ have been indicated, together with the central values $\gamma(1) = 1.237$, $\nu(1) = 0.630$ and $\theta(1) = 0.52$, on the basis of a second order DA analysis. For $N = 2$, the above cited Monte Carlo simulation of Ref.⁷⁴, yielded the estimates $a_\chi^{sc}(2) = -0.15(6)$ and $a_\xi^{sc}(2) = -0.20(4)$. Clearly, in both cases the critical parameters are slightly different from ours and this is sufficient to explain the somewhat different estimates for the correction amplitudes.

In the spin 1/2 Ising case, it has been argued long ago³¹ that $a_g^{bcc}(1)$ should be large. Recently⁴⁴, it also has been observed that, if the sc lattice Monte Carlo data of Ref.⁷⁰ are simply fitted by the function $g_r(\beta) = g^*(1 + a_g\tau^{1/2})$, the value $a_g^{sc}(1) \approx 1.13$ is obtained in fair agreement with our own estimate.

In table 7 and 8 we have listed some earlier estimates of the universal ratios $a_\xi(N)/a_\chi(N)$ and $a_g(N)/a_\chi(N)$ of correction amplitudes obtained by various methods^{20,25,36,67,77-79}. We believe that, for $N < 4$, it is not very meaningful to quote the ratios of our central estimates of a_ξ , a_χ and a_g . Indeed, as we have already pointed out, for these values of N , the amplitudes a_ξ , a_χ are small and very sensitive to the biased inputs. As a consequence, these parameters must be finely tuned, which cannot be justified until longer series will be computed. We shall indicate below a possible alternative way out of this difficulty. However, the case $N = 1$ deserves further comment. In this case, on the sc lattice, a very accurate⁸⁰ determination of $\beta_c^{sc}(1)$ is available, and also the value of $\beta_c^{bcc}(1)$ ³⁶ appears to be sufficiently safe, so that the ratios of our central estimates of the amplitudes are more trustworthy and we have reported them in parentheses.

It is interesting to recall also that, for $N = 1$, suggestions that a_g/a_χ should be large came both from earlier HT estimates⁷⁶ on the fcc lattice ($a_g^{fcc}(1)/a_\chi^{fcc}(1) \approx 3.9$) and from the RG estimates reported in Table 8. This is a further hint that the corrections to scaling should not be neglected in computing $g_r(1)$.

Unfortunately, the ϵ -expansions of these universal ratios presently only extend to second order^{77,78}, so that again we have to point out that the uncertainty of the corresponding estimates might be larger than indicated. As we have mentioned above, even the estimates of these ratios from the much longer FD expansions^{20,79} might have problems. For $N < 4$, as already observed, all series including ours are too short to accurately extract the correction amplitudes. This is particularly the case for χ_4 . Moreover, when longer series become available, our approximation procedures might need some improvement. Nevertheless these first results from HT series on an extended range of values of N seem to be qualitatively very reasonable.

As an indication of work in progress, we wish to add that, even within the present order of expansion, somewhat more accurate estimates of the critical parameters are likely to be obtained by proceeding systematically in the spirit of the Chen, Fisher, Nickel and Rehr approach^{35,36}. In the $N = 1$ case on the bcc lattice, these authors have examined HT series for families of models specified by an appropriate continuous auxiliary parameter. The members of these families interpolate between the spin 1/2 Ising and the Gaussian model and all of them are good candidates for belonging to the same universality class. (This approach easily generalizes, in various ways, to other values of N and

it is a virtue of the LCE method that the corresponding series can be derived essentially with no further computational effort.) By varying the auxiliary parameter, these authors have selected representative models such that the leading correction amplitudes a_ξ , a_χ (and a_g) vanish. It is clear that, under these conditions, even by employing ordinary unbiased DA's, the accuracy of the estimates of the universal quantities can be improved dramatically. On the other hand, within the same approach, it is probable that also the correction amplitudes will be more accurately measured by focusing on representative models in which they are sufficiently large, provided, of course, that the subleading terms are not even larger. Thus more reliable estimates might be achieved for their universal ratios, in particular in the range $N < 4$.

V. CONCLUSIONS

The main result of this paper is the extension through $O(\beta^{17})$ of the series for $\chi_4(N, \beta)$, for arbitrary N , on the sc and on the bcc lattices. Both sets of expansion coefficients have been tabulated in the appendix in order to make independent checks of their correctness and alternative analyses conveniently feasible.

A second interesting result is the numerical analysis of the critical behavior of $\chi_4(N, \beta)$ which confirms fairly well the validity of universality and hyperscaling over a wide range of values of N . We have also presented a first estimate of the size of the scaling corrections for χ , ξ^2 and χ_4 and, allowing for them, we have improved the accuracy in the determination of the critical amplitudes and of the renormalized couplings.

The agreement between our estimates of $g_r(N)$ and those from the RG approaches is generally fair, but not always perfect. At this level of approximation, it is premature to emphasize such minor discrepancies. We believe, however, that longer HT series for all quantities studied here and perhaps improved analyses are still of some interest to achieve more reliable estimates and to reduce the error bars substantially. Considering the performance of our codes, these are presently quite realistic objectives and, therefore, work is presently in progress to compute further expansion coefficients.

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TABLE I. Values for θ used in our biased evaluations and determined by FD perturbative expansion

N	0	1	2	3	4	6	8	10
θ	.478(10)	.504(8)	.529(8)	.553(12)	.573(20)	.626(10)	.670(10)	.707(10)

TABLE II. Verification of hyperscaling for various values of N .

N	Lattice	β_c^3	$\gamma + 2\Delta$	$2\gamma + 3\nu^3$
0	sc	.213493(3)	4.10(2)	4.0822(34)
	bcc	.153128(3)	4.081(8)	4.0801(34)
1	sc	.2216544(3) ⁸⁰	4.361(8)	4.3721(44)
	bcc	.157373(2) ³⁶	4.366(6)	4.3692(27)
2	sc	.45419(3)	4.665(20)	4.675(12)
	bcc	.320427(3)	4.663(15)	4.666(12)
3	sc	.69305(4)	4.953(20)	4.960(12)
	bcc	.486820(4)	4.948(15)	4.946(12)
4	sc	.93600(4)	5.24(2)	5.259(17)
	bcc	.65542(3)	5.22(2)	5.236(17)
6	sc	1.42895(6)	5.67(2)	5.691(19)
	bcc	.99644(4)	5.65(2)	5.673(17)

TABLE III. The renormalized coupling constant $g_r(N)$ for a range of values of N on the sc and the bcc lattice as obtained by various methods.

N	HT sc	HT bcc	ϵ -exp.	FD exp.	Monte Carlo
0	1.388(5)	1.387(5)	1.390(17) ⁴⁴	1.413(6) ²¹	
0				1.39 ²⁰	
1	1.408(7)	1.407(6)	1.397(8) ⁴⁴	1.411(4) ²¹	1.391(30) ⁶⁸
1	1.459(9) ⁷²			1.40 ²⁰	1.462(12) ⁶⁹
2	1.411(8)	1.411(6)	1.413(13) ⁴⁴	1.403(3) ²¹	
2				1.40 ²⁰	
3	1.409(10)	1.406(8)	1.387(7) ⁴⁴	1.391(4) ²¹	
3				1.39 ²⁰	
4	1.392(10)	1.394(10)	1.366(15) ⁴⁴	1.377(5) ²¹	
4				1.3745 ⁵⁸	
6	1.355 ⁽⁺⁵⁾ ₍₋₁₀₎	1.360 ⁽⁺⁵⁾ ₍₋₁₀₎		1.3385 ⁵⁸	
8	1.320 ⁽⁺⁸⁾ ₍₋₁₅₎	1.326 ⁽⁺⁸⁾ ₍₋₁₅₎	1.295(7) ⁴⁴	1.3045 ⁵⁸	
10	1.290 ⁽⁺⁸⁾ ₍₋₁₅₎	1.294 ⁽⁺⁸⁾ ₍₋₁₅₎		1.2745 ⁵⁸	

TABLE IV. Critical amplitudes on the sc lattice for various values of N .

N	$\beta_c(N)$	$\gamma(N)$	$\nu(N)$	$C_\chi^{sc}(N)$	$C_\xi^{sc}(N)$
0	0.213493(3)	1.1594(8)	0.5878(6)	1.115(1)	0.5101(3)
1	0.2216544(3)	1.2388(10)	0.6315(8)	1.111(1)	0.5027(3)
2	0.45419(3)	1.325(3)	0.675(2)	1.014(1)	0.4814(3)
3	0.69305(4)	1.406(3)	0.716(2)	0.9030(8)	0.4541(2)
4	0.93600(4)	1.491(4)	0.759(3)	0.7571(8)	0.4155(2)
6	1.42895(6)	1.614(5)	0.821(3)	0.6054(8)	0.3708(2)
∞		2.	1.	0.392287.. ¹⁹	0.314870.. ¹⁹

TABLE V. Critical amplitudes on the bcc lattice for various values of N .

N	$\beta_c(N)$	$\gamma(N)$	$\nu(N)$	$C_\chi^{bcc}(N)$	$C_\xi^{bcc}(N)$
0	0.153128(3)	1.1582(8)	0.5879(6)	1.087(1)	0.4846(2)
1	0.157373(2)	1.2384(6)	0.6308(5)	1.034(1)	0.4659(2)
2	0.320427(3)	1.322(3)	0.674(2)	0.918(1)	0.4371(2)
3	0.486820(4)	1.402(3)	0.714(2)	0.794(1)	0.4072(2)
4	0.65542(3)	1.484(4)	0.756(3)	0.6580(8)	0.3691(2)
6	0.99644(4)	1.608(4)	0.819(3)	0.5020(6)	0.3231(2)
∞		2.	1.	0.299741.. ¹⁹	0.263818.. ¹⁹

TABLE VI. Correction amplitudes on the sc and the bcc lattices for various values of N .

N	$a_\chi^{sc}(N)$	$a_\chi^{bcc}(N)$	$a_\xi^{sc}(N)$	$a_\xi^{bcc}(N)$	$a_g^{sc}(N)$	$a_g^{bcc}(N)$
0	-0.022(10)	-0.05(3)	-0.11(3)	-0.1	1.5(3)	1.9(4)
1	-0.10(3)	-0.08(3)	-0.12(3)	-0.08(3)	1.0(2)	1.0(2)
2	-0.04(2)	0.01(2)	-0.07(3)	-0.005(9)	0.39(8)	0.14(3)
3	0.06(3)	0.17(3)	0.003(6)	0.09(3)	0.05(10)	-0.29(6)
4	0.30(6)	0.5(1)	0.14(3)	0.28(6)	-0.12(3)	-0.55(10)
6	0.73(15)	1.1(2)	0.37(8)	0.60(15)	-0.35(8)	-0.77(20)
8	1.1(2)	1.8(3)	0.53(10)	0.92(20)	-0.43(10)	-1.0(2)

TABLE VII. The universal ratios of correction amplitudes $a_\xi(N)/a_\chi(N)$ for various values of N .

N	HT sc	HT bcc	ϵ -exp.	FD exp.
0				$0.885(50)^{20}$
1	$(1.2(4))$	$(1.0(4))$	0.65^{77}	$0.762(30)^{20}$
1		$0.70(3)^{76}; 0.85(5)^{36}$	$0.56(15)^{25}$	$0.65(5)^{79}$
2			0.63^{77}	$0.687(10)^{20}$
2				$0.615(5)^{79}$
3				$0.637(5)^{20}$
3				0.60^{79}
4	$0.49(15)$	$0.55(15)$		
6	$0.51(15)$	$0.55(15)$		
8	$0.49(15)$	$0.54(15)$		

 TABLE VIII. The universal ratios of correction amplitudes $a_g(N)/a_\chi(N)$ for various values of N .

N	HT sc	HT bcc	ϵ -exp.	FD exp.
0				
1	$(-3.0(5))$	$(-3.5(5))$	$-2.2(5)^{67}$	$-2.85(6)^{79}$
2				$-2.08(5)^{79}$
3				$-1.65(4)^{79}$
4				
6	$-1.2(4)$	$-0.7(2)$		
8	$-1.1(4)$	$0.8(3)$		

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APPENDIX A: THE SECOND FIELD DERIVATIVE OF THE SUSCEPTIBILITY ON THE SC LATTICE

The HT expansion coefficients of the second field derivative of the susceptibility

$$\chi_4(N, \beta) = \frac{3N}{N+2} \sum_{x,y,z} \langle v(0) \cdot v(x) v(y) \cdot v(z) \rangle_c = \frac{3N}{N+2} \left(-\frac{2}{N} + \sum_{r=1}^{\infty} d_r(N) \beta^r \right)$$

on the sc lattice are

$$d_1(N) = -48/N^2$$

$$d_2(N) = \frac{-1248 - 660 N}{N^3 (2 + N)}$$

$$d_3(N) = \frac{-12480 - 6912 N}{N^4 (2 + N)}$$

$$d_4(N) = \frac{-851712 - 1128192 N - 474000 N^2 - 61236 N^3}{N^5 (2 + N)^2 (4 + N)}$$

$$d_5(N) = \frac{-6573312 - 8786880 N - 3725856 N^2 - 483840 N^3}{N^6 (2 + N)^2 (4 + N)}$$

$$d_6(N) = \frac{-565908480 - 1137490944 N - 877991616 N^2 - 321566208 N^3 - 55124016 N^4 - 3514968 N^5}{N^7 (2 + N)^3 (4 + N) (6 + N)}$$

For the coefficients which follow it is typographically more convenient to set $d_r(N) = P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$,

$$P_7(N) = -3849744384 - 7739114496 N - 5976661248 N^2 - 2190260352 N^3 - 375495552 N^4 - 23938560 N^5$$

$$Q_7(N) = N^8 (2 + N)^3 (4 + N) (6 + N)$$

$$P_8(N) = -1607361822720 - 4630934495232 N - 5658731108352 N^2 - 3815032300032 N^3 - 1545703906176 N^4$$

$$-383951922240 N^5 - 56942341632 N^6 - 4600824312 N^7 - 154867284 N^8$$

$$Q_8(N) = N^9 (2 + N)^4 (4 + N)^2 (6 + N) (8 + N)$$

$$P_9(N) = -10146634334208 - 29145299066880 N - 35515117682688 N^2 - 23883563143680 N^3 - 9654774400512 N^4$$

$$-2393329321728 N^5 - 354292592640 N^6 - 28580124768 N^7 - 960719616 N^8$$

$$Q_9(N) = N^{10} (2 + N)^4 (4 + N)^2 (6 + N) (8 + N)$$

$$P_{10}(N) = -4986778413957120 - 18502905604472832 N - 30434129019666432 N^2 - 29229711376023552 N^3$$

$$-18173047179460608 N^4 - 7663189901412864 N^5 - 2231748901824768 N^6 - 448039233434880 N^7$$

$$-60661040264064 N^8 - 5267106682272 N^9 - 263609235360 N^{10} - 5754914568 N^{11},$$

$$Q_{10}(N) = N^{11} (2 + N)^5 (4 + N)^3 (6 + N) (8 + N) (10 + N)$$

$$P_{11}(N) = -29957292231229440 - 110697821461807104 N - 181359443339575296 N^2 - 173526582721855488 N^3$$

$$-107505154356572160 N^4 - 45183600633858048 N^5 - 13119088167641088 N^6 - 2626537796155392 N^7$$

$$-354741723247872 N^8 - 30735239578752 N^9 - 1535369868288 N^{10} - 33465664512 N^{11}$$

$$Q_{11}(N) = N^{12} (2 + N)^5 (4 + N)^3 (6 + N) (8 + N) (10 + N)$$

$$P_{12}(N) = -25424963775485706240 - 112568160566969892864 N - 226122593284806672384 N^2 - 272739288108751650816 N^3$$

$$-220335024310830366720 N^4 - 125926002861175824384 N^5 - 52429589825842464768 N^6 - 16133360608103531520 N^7$$

$$-3682631735427354624 N^8 - 619886922488017920 N^9 - 75677317494258048 N^{10} - 6492816409650048 N^{11}$$

$$-369850295426784 N^{12} - 12514445149200 N^{13} - 189708636600 N^{14}$$

$$Q_{12}(N) = N^{13} (2 + N)^6 (4 + N)^3 (6 + N)^2 (8 + N) (10 + N) (12 + N)$$

$$P_{13}(N) = -147436255220890337280 - 649808498848293715968 N - 1299512129528339103744 N^2$$

$$-1560658843095801004032 N^3 - 1255571803469692796928 N^4 - 714760376636548620288 N^5$$

$$-296490805364682670080 N^6 - 90920656735938183168 N^7 - 20688048472922093568 N^8$$

$$-3472336842523894272 N^9 - 422813491673485824 N^{10} - 36192084771724800 N^{11}$$

$$-2057419542670080 N^{12} - 69492377025984 N^{13} - 1051829121024 N^{14}$$

$$Q_{13}(N) = N^{14} (2 + N)^6 (4 + N)^3 (6 + N)^2 (8 + N) (10 + N) (12 + N)$$

$$P_{14}(N) = -566463587862368653148160 - 3043875242905724409348096 N - 7576777562322913437155328 N^2$$

$$-11598723404301679271608320 N^3 - 12226130280460910006894592 N^4 - 9415725334597698602139648 N^5$$

$$-5486072371880331913592832 N^6 - 2470783134534602565992448 N^7 - 871335330057590064906240 N^8$$

$$-242243678099334349271040 N^9 - 53184931154543324258304 N^{10} - 9193953185463740998656 N^{11}$$

$$-1241394935192535991296 N^{12} - 129075746454058556160 N^{13} - 10102244514482628864 N^{14}$$

$$-573981185124034560 N^{15} - 22283021804865984 N^{16} - 527221923353952 N^{17} - 5719330613520 N^{18}$$

$$Q_{14}(N) = N^{15} (2 + N)^7 (4 + N)^4 (6 + N)^3 (8 + N) (10 + N) (12 + N) (14 + N)$$

$$P_{15}(N) = -3199047998804219563868160 - 17113814040741606825394176 N - 42413540836361700246552576 N^2$$

$$-64650199580585358413266944 N^3 - 67863921600679932441133056 N^4 - 52054127611661734870253568 N^5$$

$$\begin{aligned}
& -30212418901698609794777088 N^6 - 13556920252810313722822656 N^7 - 4764324172851484585525248 N^8 \\
& -1320233119260810969808896 N^9 - 288978328415607033544704 N^{10} - 49814490389339231121408 N^{11} \\
& -6708703760313406396416 N^{12} - 695898456216849782784 N^{13} - 54348379646212915200 N^{14} \\
& -3081948095561442816 N^{15} - 119439454925014272 N^{16} - 2821599683684352 N^{17} - 30566943406080 N^{18}
\end{aligned}$$

$$Q_{15}(N) = N^{16} (2 + N)^7 (4 + N)^4 (6 + N)^3 (8 + N) (10 + N) (12 + N) (14 + N)$$

$$P_{16}(N) = -18299002328234221207226941440 - 114585264379209072949397028864 N - 336677952680726802363970486272 N^2$$

$$\begin{aligned}
& -617145013898356017378109685760 N^3 - 791521185299488842160294330368 N^4 \\
& -755125087023410853999229796352 N^5 - 556163233950005337533492232192 N^6 \\
& -323972815609871396650371514368 N^7 - 151693834060073149464108859392 N^8 \\
& -57712428469432137886012145664 N^9 - 17963030647495383412208861184 N^{10} \\
& -4590796640925368216011948032 N^{11} - 964233613788142396008136704 N^{12} \\
& -166150396381094082218360832 N^{13} - 23381941196613066788198400 N^{14} \\
& -2666695999548097875723264 N^{15} - 243616432556930571577344 N^{16} \\
& -17525072840959462713984 N^{17} - 968152139098787129664 N^{18} - 39538186077707686464 N^{19} \\
& -1121781497993998176 N^{20} - 19697772998733576 N^{21} - 160868281485204 N^{22}
\end{aligned}$$

$$Q_{16}(N) = N^{17} (2 + N)^8 (4 + N)^5 (6 + N)^3 (8 + N)^2 (10 + N) (12 + N) (14 + N) (16 + N)$$

$$P_{17}(N) = -101235958788462673253105664000 - 631215419419149274713371443200 N - 1846824461720939657841963171840 N^2$$

$$\begin{aligned}
& -3371230360792179922631399571456 N^3 - 4306156546428529311715461955584 N^4 \\
& -4091812122737075444969361113088 N^5 - 3002060652157587317256634761216 N^6 \\
& -1742224189122069022527046287360 N^7 - 812841546356917149508011294720 N^8 \\
& -308190677384300600694770368512 N^9 - 95612813294752704503595663360 N^{10} \\
& -24360624222950336197451317248 N^{11} - 5101833629477951341726236672 N^{12} \\
& -876738435402445320757764096 N^{13} - 123070791802592404991434752 N^{14} \\
& -14003418390237375888211968 N^{15} - 1276532972320935406141440 N^{16} \\
& -91648782429013822778880 N^{17} - 5053887369689137387008 N^{18} - 206055475899900727296 N^{19}
\end{aligned}$$

$$-5837513893110173568 N^{20} - 102365100472039776 N^{21} - 834986039880960 N^{22}$$

$$Q_{16}(N) = N^{18} (2 + N)^8 (4 + N)^5 (6 + N)^3 (8 + N)^2 (10 + N) (12 + N) (14 + N) (16 + N)$$

In particular for $N = 0$ we have (in terms of the variable $\tilde{\beta} = \beta/N$)

$$\begin{aligned} \hat{\chi}_4(\tilde{\beta}) = & -3 - 72\tilde{\beta} - 936\tilde{\beta}^2 - 9360\tilde{\beta}^3 - 79848\tilde{\beta}^4 - 616248\tilde{\beta}^5 - 4421160\tilde{\beta}^6 - 30076128\tilde{\beta}^7 - 196211160\tilde{\beta}^8 \\ & -1238602824\tilde{\beta}^9 - 7609219992\tilde{\beta}^{10} - 45711200304\tilde{\beta}^{11} - 269412610536\tilde{\beta}^{12} - 1562290776792\tilde{\beta}^{13} - 8932238341992\tilde{\beta}^{14} \\ & -50443946980992\tilde{\beta}^{15} - 281783630311272\tilde{\beta}^{16} - 1558917555928200\tilde{\beta}^{17} \end{aligned}$$

For $N = 1$ [the spin 1/2 Ising model], we have

$$\begin{aligned} \chi_4(1, \beta) = & -2 - 48\beta - 636\beta^2 - 6464\beta^3 - 55892\beta^4 - 2174432/5\beta^5 - 47009464/15\beta^6 - 2239468288/105\beta^7 - 14570710772/105\beta^8 \\ & -823130010272/945\beta^9 - 25080975789304/4725\beta^{10} - 1640401398782848/51975\beta^{11} - 28654566671774104/155925\beta^{12} \\ & -2130434175575247424/2027025\beta^{13} - 83969257269976828688/14189175\beta^{14} - 6995762565293277161216/212837625\beta^{15} \\ & -38389375874347206695732/212837625\beta^{16} - 272537955948789968719904/278326125\beta^{17} \dots \end{aligned}$$

For $N = 2$ [the XY model] we have

$$\begin{aligned} \chi_4(2, \beta) = & -3/2 - 18\beta - 963/8\beta^2 - 1233/2\beta^3 - 171687/64\beta^4 - 167661/16\beta^5 - 38749413/1024\beta^6 - 32973957/256\beta^7 \\ & -2142639141/5120\beta^8 - 13411622379/10240\beta^9 - 3907085119879/983040\beta^{10} - 240713424017/20480\beta^{11} \\ & -1247905418479081/36700160\beta^{12} - 166057186013983/1720320\beta^{13} \\ & -407002859073704999/1509949440\beta^{14} - 1960425200264079271/2642411520\beta^{15} \\ & -3835682132124206551811/1902536294400\beta^{16} - 245360122597207497559/45298483200\beta^{17} \dots \end{aligned}$$

For $N = 3$ [the Heisenberg classical model], we have

$$\begin{aligned} \chi_4(3, \beta) = & -6/5 - 48/5\beta - 1076/25\beta^2 - 11072/75\beta^3 - 677044/1575\beta^4 - 981856/875\beta^5 - 958584296/354375\beta^6 \\ & -261075968/42525\beta^7 - 362572843588/27286875\beta^8 - 9268612328224/334884375\beta^9 \\ & -3647348945492264/65302453125\beta^{10} - 1655707479102099328/15084866671875\beta^{11} \\ & -3670221101428789064/17405615390625\beta^{12} - 53935317813474946624/135763800046875\beta^{13} \\ & -99068754350666844524336/134632435046484375\beta^{14} - 4880947680478330092600064/3635075746255078125\beta^{15} \\ & -192879202499123356385626829692/79771737251567689453125\beta^{16} \\ & -38137398242459901685390609504/8863526361285298828125\beta^{17} \dots \end{aligned}$$

APPENDIX B: THE SECOND FIELD DERIVATIVE OF THE SUSCEPTIBILITY ON THE BCC LATTICE

The HT expansion coefficients of the second field derivative of the susceptibility on the bcc lattice are

$$d_1(N) = \frac{-64}{N^2}$$

$$d_2(N) = \frac{-2304 - 1200 N}{N^3 (2 + N)}$$

$$d_3(N) = \frac{-32256 - 17408 N}{N^4 (2 + N)}$$

$$d_4(N) = \frac{-3086848 - 4038528 N - 1679424 N^2 - 215600 N^3}{N^5 (2 + N)^2 (4 + N)}$$

$$d_5(N) = \frac{-33383424 - 44140288 N - 18541952 N^2 - 2396160 N^3}{N^6 (2 + N)^2 (4 + N)}$$

$$d_6(N) = \frac{-4025769984 - 8048769024 N - 6183804416 N^2 - 2256738944 N^3 - 386061056 N^4 - 24592512 N^5}{N^7 (2 + N)^3 (4 + N) (6 + N)}$$

For the coefficients which follow it is typographically more convenient to set $d_r(N) = P_r(N)/Q_r(N)$ and to tabulate separately the numerator polynomial $P_r(N)$ and the denominator polynomial $Q_r(N)$,

$$P_7(N) = -38338166784 - 76883050496 N - 59262616576 N^2 - 21695712768 N^3 - 3720514560 N^4 - 237404160 N^5$$

$$Q_7(N) = N^8 (2 + N)^3 (4 + N) (6 + N)$$

$$P_8(N) = -22398548705280 - 64579066675200 N - 78988001214464 N^2 - 53316161983488 N^3 -$$

$$21632718536704 N^4 - 5382539503872 N^5 - 799715671040 N^6 - 64733334688 N^7 - 2182679664 N^8$$

$$Q_8(N) = N^9 (2 + N)^4 (4 + N)^2 (6 + N) (8 + N)$$

$$P_9(N) = -197760671023104 - 570072979439616 N - 697267355156480 N^2 - 470715808290816 N^3$$

$$-191030743859200 N^4 - 47540867562496 N^5 - 7064458432512 N^6 - 571883674496 N^7 - 19283466240 N^8$$

$$Q_9(N) = N^{10} (2 + N)^4 (4 + N)^2 (6 + N) (8 + N)$$

$$P_{10}(N) = -135893648670720000 - 506812733453762560 N - 837961375615287296 N^2$$

$$-808982617436258304 N^3 - 505548566324297728 N^4 - 214244480842262528 N^5 - 62694667322504192 N^6$$

$$-12643889177861632 N^7 - 1719173758968320 N^8 - 149853167848320 N^9$$

$$-7525864311296 N^{10} - 164795456384 N^{11}$$

$$Q_{10}(N) = N^{11} (2 + N)^5 (4 + N)^3 (6 + N) (8 + N) (10 + N)$$

$$P_{11}(N) = -1141044912712581120 - 4249040663391240192 N - 7015555079356284928 N^2$$

$$-6764393828229578752 N^3 - 4222404584768831488 N^4 - 1787577880615559168 N^5$$

$$-522623903804669952 N^6 - 105313027917844480 N^7 - 14308686372335616 N^8$$

$$-1246403878745600 N^9 - 62559888474112 N^{10} - 1369193398272 N^{11}$$

$$Q_{11}(N) = N^{12} (2 + N)^5 (4 + N)^3 (6 + N) (8 + N) (10 + N)$$

$$P_{12}(N) = -1353224519826548981760 - 6043648174007450075136 N - 12246178519860941684736 N^2$$

$$-14898334498567895384064 N^3 - 12137516326724153901056 N^4 - 6993670039948027871232 N^5$$

$$-2934699277523969310720 N^6 - 909774043355138494464 N^7 - 209115782694899085312 N^8$$

$$-35427294974940029952 N^9 - 4350609416485181440 N^{10} - 375257673683951872 N^{11}$$

$$-21477632431608320 N^{12} - 729782591023616 N^{13} - 11103491816320 N^{14}$$

$$Q_{12}(N) = N^{13} (2 + N)^6 (4 + N)^3 (6 + N)^2 (8 + N) (10 + N) (12 + N)$$

$$P_{13}(N) = -10962814317012556185600 - 48855640313305403228160 N - 98790788652478570168320 N^2$$

$$-119949033747113622634496 N^3 - 97539319548016125280256 N^4 - 56104431660229407899648 N^5$$

$$-23504510497602793209856 N^6 - 7275637584279889346560 N^7 - 1670046301220816199680 N^8$$

$$-282577439283542464512 N^9 - 34662501972894392320 N^{10} - 2986763073027756032 N^{11}$$

$$-170792230703923200 N^{12} - 5798723600550656 N^{13} - 88165648564224 N^{14}$$

$$Q_{13}(N) = N^{14} (2 + N)^6 (4 + N)^3 (6 + N)^2 (8 + N) (10 + N) (12 + N)$$

$$P_{14}(N) = -58832199553811357682892800 - 319656261223016276133150720 N$$

$$-804506884372627522672656384 N^2 - 1245051701802465742955741184 N^3$$

$$-1326506792510663276202295296 N^4 - 1032281846527637618727845888 N^5$$

$$-607550530706546656156057600 N^6 - 276285749123163236534124544 N^7$$

$$-98336895868270792502870016 N^8 - 27578892092469750017228800 N^9$$

$$-6104887252432902432088064 N^{10} - 1063450425127326143963136 N^{11}$$

$$-144612518832699900119040 N^{12} - 15134717223692504156160 N^{13}$$

$$-1191613756119982431744 N^{14} - 68071027536755782912 N^{15}$$

$$-2655538734547345536 N^{16} - 63104921069570176 N^{17} - 687220401024000 N^{18}$$

$$Q_{14}(N) = N^{15} (2 + N)^7 (4 + N)^4 (6 + N)^3 (8 + N) (10 + N) (12 + N) (14 + N)$$

$$P_{15}(N) = -463996271577436867233054720 - 2515152883396655909638766592 N - 6315643027876160842940547072 N^2$$

$$-9752398559014235653061738496 N^3 - 10368212405864877478064947200 N^4$$

$$-8051953080310134001281007616 N^5 - 4729721489677266022278627328 N^6$$

$$-2146872892026044965004247040 N^7 - 762790924816930341473615872 N^8$$

$$-213577145756373531731427328 N^9 - 47205533784466451589431296 N^{10}$$

$$-8211418569550233134678016 N^{11} - 1115168266958340559364096 N^{12}$$

$$-116570698649671407411200 N^{13} - 9168049245864669396992 N^{14}$$

$$-523207897673566349312 N^{15} - 20392833008810153984 N^{16} - 484218500284702720 N^{17} - 5269451094097920 N^{18}$$

$$Q_{15}(N) = N^{16} (2 + N)^7 (4 + N)^4 (6 + N)^3 (8 + N) (10 + N) (12 + N) (14 + N)$$

$$P_{16}(N) = -3706032155932253013818065551360 - 23511532957902177899390315790336 N$$

$$-69984745734719724031602453381120 N^2 - 129942518358149653545095685734400 N^3$$

$$-168777163612514463301490677121024 N^4 - 163020601132834974169821018587136 N^5$$

$$-121523005151702351276069798019072 N^6 - 71619975067826655255089882595328 N^7$$

$$-33913954817349955356957471670272 N^8 - 13042529099485925377885106077696 N^9$$

$$-4101432698345456358253122813952 N^{10} - 1058465798499513773553711382528 N^{11}$$

$$-224369655134353441277932077056 N^{12} - 38996967102960612063728975872 N^{13}$$

$$-5532347732173533521822597120 N^{14} - 635702582117166696141393920 N^{15}$$

$$-58478148624089490157039616 N^{16} - 4233637928944288589514240 N^{17}$$

$$-235251240860669046267648 N^{18} - 9658642370970465608960 N^{19}$$

$$-275363031292904868480 N^{20} - 4856360702431679584 N^{21} - 39817256443223280 N^{22}$$

$$Q_{16}(N) = N^{17} (2 + N)^8 (4 + N)^5 (6 + N)^3 (8 + N)^2 (10 + N) (12 + N) (14 + N) (16 + N)$$

$$P_{17}(N) = -28625170809317622639273757900800 - 181166520457100717483475767132160 N$$

$$-537991814958099333262399103827968 N^2 - 996597299051903297408062545985536 N^3$$

$$-1291524895757272450924715899355136 N^4 - 1244744825531489741899730562056192 N^5$$

$$\begin{aligned}
& -925926421634294684791564008423424 N^6 - 544586262055545477273422842036224 N^7 \\
& -257372711584126360801923714514944 N^8 - 98795363631395088253841625841664 N^9 \\
& -31012859792452238511372108300288 N^{10} - 7990178234700265757218112274432 N^{11} \\
& -1691062951010009264622059913216 N^{12} - 293484614468719284469043789824 N^{13} \\
& -41578182122239994876120399872 N^{14} - 4771486216282518354479939584 N^{15} \\
& -43840783425226343769522176 N^{16} - 31704698511771193169897472 N^{17} \\
& -1759972842290831947573248 N^{18} - 72192427708156325499904 N^{19} \\
& -2056451245211192506368 N^{20} - 36240554109317191808 N^{21} - 296932880108021760 N^{22} \\
& Q_{17}(N) = N^{18} (2+N)^8 (4+N)^5 (6+N)^3 (8+N)^2 (10+N) (12+N) (14+N) (16+N)
\end{aligned}$$

In particular for $N = 0$, we have (in terms of the variable $\tilde{\beta} = \beta/N$)

$$\begin{aligned}
\chi_4(\tilde{\beta}) = & -3 - 96\tilde{\beta} - 1728\tilde{\beta}^2 - 24192\tilde{\beta}^3 - 289392\tilde{\beta}^4 - 3129696\tilde{\beta}^5 - 31451328\tilde{\beta}^6 - 299516928\tilde{\beta}^7 - 2734197840\tilde{\beta}^8 \\
& -24140706912\tilde{\beta}^9 - 207357252000\tilde{\beta}^{10} - 1741096363392\tilde{\beta}^{11} - 14339282987664\tilde{\beta}^{12} - 116166160551840\tilde{\beta}^{13} \\
& -927691099407360\tilde{\beta}^{14} - 7316490196952064\tilde{\beta}^{15} - 57068641022992368\tilde{\beta}^{16} - 440794771444139040\tilde{\beta}^{17}
\end{aligned}$$

For $N = 1$ [the spin 1/2 Ising model], we have

$$\begin{aligned}
\chi_4(1, \beta) = & -2 - 64\beta - 1168\beta^2 - 49664/3\beta^3 - 601360/3\beta^4 - 32820608/15\beta^5 - 996463616/45\beta^6 - 66712488448/315\beta^7 \\
& -122056132496/63\beta^8 - 48489867797888/2835\beta^9 - 2078558044733696/14175\beta^{10} \\
& -191285725186144768/155925\beta^{11} - 188087379936809600/18711\beta^{12} - 492034524872707515136/6081075\beta^{13} \\
& -27296494302637993572352/42567525\beta^{14} - 3201197677867739316248576/638512875\beta^{15} \\
& -4945781553886665074906384/127702575\beta^{16} - 3212941768987291424807915648/10854718875\beta^{17}
\end{aligned}$$

For $N = 2$ [the XY model], we have

$$\begin{aligned}
\chi_4(2, \beta) = & -3/2 - 24\beta - 441/2\beta^2 - 1572\beta^3 - 153175/16\beta^4 - 104981/2\beta^5 - 68238647/256\beta^6 \\
& -40884131/32\beta^7 - 1404217891/240\beta^8 - 66125311269/2560\beta^9 - 16313466298147/147456\beta^{10} \\
& -85154694896333/184320\beta^{11} - 51968444431571323/27525120\beta^{12} - 156353752523792639/20643840\beta^{13} \\
& -236982809408746803649/7927234560\beta^{14} - 400259785750849937/3440640\beta^{15} \\
& -637876955227851294690449/1426902220800\beta^{16} - 48476858619011264835409/28538044416\beta^{17} \dots
\end{aligned}$$

For $N = 3$ [the Heisenberg classical model], we have

$$\begin{aligned}
\chi_4(3, \beta) = & -6/5 - 64/5\beta - 1968/25\beta^2 - 5632/15\beta^3 - 36138448/23625\beta^4 - 132459392/23625\beta^5 - 1348186624/70875\beta^6 \\
& -194131778048/3189375\beta^7 - 228357648983536/1227909375\beta^8 - 2016513048715136/3683728125\beta^9 \\
& -3366834959446902016/2154980953125\beta^{10} - 1872724144398680576/430996190625\beta^{11} \\
& -2674610910995288182912/226273000078125\beta^{12} - 367131807235133274368/11636897146875\beta^{13} \\
& -100350171877191930199645952/1211691915418359375\beta^{14} - 778819344453674012213276672/3635075746255078125\beta^{15} \\
& -130937251648404408278387043967856/239315211754703068359375\beta^{16} \\
& -22035313880683966424477522278528/15954347450313537890625\beta^{17}
\end{aligned}$$